

# A short note on a conjecture of Okounkov about a $q$ -analogue of multiple zeta values

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## Abstract

In [Ok] Okounkov studies a specific  $q$ -analogue of multiple zeta values and makes some conjectures on their algebraic structure. In this note we want to compare Okounkov's  $q$ -analogues to the generating function for multiple divisor sums defined in [BK].

## 1 Introduction

Multiple zeta values are natural generalizations of the Riemann zeta values that are defined for integers  $s_1 > 1$  and  $s_i \geq 1$  for  $i > 1$  by

$$\zeta(s_1, \dots, s_l) := \sum_{n_1 > n_2 > \dots > n_l > 0} \frac{1}{n_1^{s_1} \dots n_l^{s_l}}.$$

Because of its occurrence in various fields of mathematics and physics these real numbers are of particular interest. In [Ok] Okounkov discusses a conjectural connection from enumerative geometry of some Hilbert schemes to a specific  $q$ -analogue  $Z(s_1, \dots, s_l)$  of the multiple zeta-values. He denotes by  $\mathbf{qMZV}$  the  $\mathbb{Q}$ -algebra generated by these. In this short note we want to discuss the connection of these  $q$ -multiple zeta values to the algebra  $\mathcal{MD}$  of generating functions for multiple divisor sums  $[s_1, \dots, s_l]$  defined by the authors in [BK]. More precisely we have

**Theorem 1.1.** Let  $\mathcal{MD}^\# = \langle [s_1, \dots, s_l] \in \mathcal{MD} \mid s_i > 1 \forall i \text{ or } s_1 = \emptyset \rangle_{\mathbb{Q}}$ .

- i) The sub vector space  $\mathcal{MD}^\#$  is in fact a sub algebra of  $\mathcal{MD}$ .
- ii) We have  $\mathbf{qMZV} = \mathcal{MD}^\#$ , in particular the  $\mathbb{Q}$ -vector space generated by the  $Z(s_1, \dots, s_l)$  is closed under multiplication.
- iii) We have  $q \frac{d}{dq} Z(k) \in \mathbf{qMZV}$  for all  $k \geq 2$ .

The first two statements are merely a reformulation results implicitly contained in [BK]. The third is direct consequence of some explicit formula given in [BK]. It gives some evidence to the conjecture of Okounkov, that the operator  $d$  is a derivation on  $\mathbf{qMZV}$ .

## 2 $q$ -analogues of multiple zeta values

In the following we fix a subset  $S \subset \mathbb{N}$ , which we consider as the support for index entries, i.e. we assume  $s_1, \dots, s_l \in S$ . For each  $s \in S$  we let  $Q_s(t) \in \mathbb{Q}[t]$  be a polynomial with  $Q_s(0) = 0$  and  $Q_s(1) \neq 0$ . We set  $Q = \{Q_s(t)\}_{s \in S}$ . A sum of the form

$$Z_Q(s_1, \dots, s_l) := \sum_{n_1 > \dots > n_l > 0} \prod_{j=1}^l \frac{Q_{s_j}(q^{n_j})}{(1 - q^{n_j})^{s_j}} \quad (2.1)$$

with polynomials  $Q_s$  as before, defines a  $q$ -analogue of a multiple zeta-value of weight  $k = s_1 + \dots + s_l$  and length  $l$ . Observe only because of  $Q_{s_1}(0) = 0$  this defines an element of  $\mathbb{Q}[[q]]$ . This notion is due to the identity

$$\begin{aligned} \lim_{q \rightarrow 1} (1 - q)^k Z_Q(s_1, \dots, s_l) &= \sum_{n_1 > \dots > n_l > 0} \prod_{j=1}^l \lim_{q \rightarrow 1} \left( Q_{s_j}(q^{n_j}) \frac{(1 - q)^{s_j}}{(1 - q^{n_j})^{s_j}} \right) \\ &= Q_{s_1}(1) \dots Q_{s_l}(1) \cdot \zeta(s_1, \dots, s_l). \end{aligned}$$

Here we used that  $\lim_{q \rightarrow 1} (1 - q)^s / (1 - q^n)^s = 1/n^s$  and with the same arguments as in [BK] Proposition 6.4, the above identity can be justified for all  $(s_1, \dots, s_l)$  with  $s_1 > 1$ . Related definition for  $q$ -analogues of multiple zeta values are given in [Br], [Ta], [Zu] and [OOZ]. It is convenient to define  $Z_Q(\emptyset) = 1$  and then we denote the vector space spanned by all these elements by

$$Z(Q, S) := \langle Z_Q(s_1, \dots, s_l) \mid l \geq 0 \text{ and } s_1, \dots, s_l \in S \rangle_{\mathbb{Q}}. \quad (2.2)$$

Note by the above convention we have that  $\mathbb{Q}$  is contained in this space.

**Lemma 2.1.** If for each  $r, s \in S$  there exists numbers  $\lambda_j(r, s) \in \mathbb{Q}$  such that

$$Q_r(t) \cdot Q_s(t) = \sum_{\substack{j \in S \\ 1 \leq j \leq r+s}} \lambda_j(r, s) (1 - t)^{r+s-j} Q_j(t), \quad (2.3)$$

then the vector space  $Z(Q, S)$  is a  $\mathbb{Q}$ -algebra,

**Proof.** We have to show that  $Z_Q(s_1, \dots, s_l) \cdot Z_Q(r_1, \dots, r_m) \in Z(Q, S)$  and illustrate this in the  $l = m = 1$  case because the higher length case will be clear after this. Suppose there is a representation of the form (2.3) then it is

$$\begin{aligned} Z_Q(r) \cdot Z_Q(s) &= \sum_{n_1 > 0} \frac{Q_r(q^{n_1})}{(1 - q^{n_1})^r} \cdot \sum_{n_2 > 0} \frac{Q_s(q^{n_2})}{(1 - q^{n_2})^s} \\ &= \sum_{n_1 > n_2 > 0} \dots + \sum_{n_2 > n_1 > 0} \dots + \sum_{n_1 = n_2 = n > 0} \frac{Q_r(q^n) Q_s(q^n)}{(1 - q^n)^{r+s}} \\ &= Z_Q(r, s) + Z_Q(s, r) + \sum_{j \in S'} \lambda_j Z_Q(j) \in Z(S, Q). \end{aligned}$$

□

We give three examples of  $q$ -analogues of multiple zeta values, which are currently considered by different authors where just the second and the third will be of interest in the rest of this note.

- 0) The polynomials  $Q_s^T(t) = t^{s-1}$  are considered in [Ta] and sums of the form (2.1) with  $s_1 > 1$  and  $s_2, \dots, s_l \geq 1$  are studied there.
- i) In [BK] the authors choose  $Q_s^E(t) = \frac{1}{(s-1)!} t P_{s-1}(t)$ , where the  $P_s(t)$  are the eulerian polynomials defined by

$$\frac{t P_{s-1}(t)}{(1-t)^s} = \sum_{d=1}^{\infty} d^{s-1} t^d$$

for  $s \geq 0$ . With this define for all  $s_1, \dots, s_l \in \mathbb{N}$

$$[s_1, \dots, s_l] := \sum_{n_1 > \dots > n_l > 0} \prod_{j=1}^l \frac{Q_{s_j}^E(q^{n_j})}{(1 - q^{n_j})^{s_j}}.$$

and set

$$\mathcal{MD} = Z(\{Q_s^E(t)\}_s, \mathbb{N}).$$

These *brackets* are generating functions for multiple divisor sums and they occur in the Fourier expansion of multiple Eisenstein series.

- ii) Okounkov chooses the following polynomials in [Ok]

$$Q_s^O(t) = \begin{cases} t^{\frac{s}{2}} & s = 2, 4, 6, \dots \\ t^{\frac{s-1}{2}}(1+t) & s = 3, 5, 7, \dots \end{cases}$$

and defines for  $s_1, \dots, s_l \in S = \mathbb{N}_{>1}$

$$Z(s) = \sum_{n_1 > \dots > n_l > 0} \prod_{j=0}^l \frac{Q_{s_j}^O(q^{n_j})}{(1 - q^{n_j})^{s_j}}.$$

We write for the space of the Okounkov  $q$ -multiple zetas

$$\mathfrak{qMZV} = Z(\{Q_s^O(t)\}_s, \mathbb{N}_{>1}).$$

**Proposition 2.2.** For the polynomials above we have

- i) for  $r, s \in \mathbb{N}$  and  $Q_j^E(t) = \frac{1}{(s-1)!} t P_{s-1}(t)$

$$Q_r^E(t) \cdot Q_s^E(t) = \sum_{j=1}^r \lambda_{r,s}^j (1-t)^{r+s-j} Q_j^E(t) + \sum_{j=1}^s \lambda_{s,r}^j (1-t)^{r+s-j} Q_j^E(t) + Q_{r+s}^E(t),$$

where the coefficient  $\lambda_{a,b}^j \in \mathbb{Q}$  for  $1 \leq j \leq a$  is given by

$$\lambda_{a,b}^j = (-1)^{b-1} \binom{a+b-j-1}{a-j} \frac{B_{a+b-j}}{(a+b-j)!}.$$

ii) for  $r, s \in \mathbb{N}_{>1}$  it is

$$Q_r^O(t) \cdot Q_s^O(t) = \begin{cases} Q_{r+s}^O(X) & , r + s \text{ even} \\ 2Q_{r+s}^O(t) + (1-t)^2 Q_{r+s-2}^O(t) & , r + s \text{ odd} . \end{cases}$$

In particular, because of Lemma 2.1, the vector spaces  $\mathcal{MD}$  and  $\mathbf{qMZV}$  are  $\mathbb{Q}$ -algebras.

**Proof.** In [BK] the claim i) is proven. The cases in ii) are checked easily.  $\square$

**Corollary 2.3.**  $\mathcal{MD}^\# = Z(\{Q_s^E\}_s, \mathbb{N}_{>1})$  is a sub algebra of  $\mathcal{MD}$ .

*Proof.* Using Proposition 2.2 it is easy to see that it suffices to show that

$$\lambda_{a,b}^1 + \lambda_{b,a}^1 = ((-1)^{a-1} + (-1)^{b-1}) \binom{a+b-2}{a-1} \frac{B_{a+b-1}}{(a+b-1)!}$$

vanishes for  $a, b > 1$ . This term clearly vanishes when  $a$  and  $b$  have different parity. In the other case  $a+b-1$  is odd and greater than 1, as  $a, b > 1$ . It is well known that in this case  $B_{a+b-1} = 0$ , from which we deduce that  $\lambda_{a,b}^1 + \lambda_{b,a}^1 = 0$ .  $\square$

**Theorem 2.4.** Let  $Z(Q, \mathbb{N}_{>1})$  be any family of q-analogues of multiple zeta values as in (2.2), where each  $Q_s(t) \in Q$  is a polynomial with degree at most  $s-1$ , then

$$Z(Q, \mathbb{N}_{>1}) = \mathcal{MD}^\# .$$

and therefore all such families of q-analogues of multiple zeta values are  $\mathbb{Q}$ -subalgebras of  $\mathcal{MD}$ . In particular  $\mathbf{qMZV} = \mathcal{MD}^\#$ .

**Proof.** To proof the first equality it is sufficient to show that for each  $s > 1$  there are numbers  $\lambda_j \in \mathbb{Q}$  with  $2 \leq j \leq s$  such that

$$\frac{Q_s(t)}{(1-t)^s} = \sum_{j=2}^s \lambda_j \frac{Q_j^E(t)}{(1-t)^j} .$$

The space of polynomials with at most degree  $s-1$  and no constant term has dimension  $s-1$ . For  $2 \leq j \leq s$  the polynomials  $(1-t)^{s-j} Q_j'(t)$  are all linear independent since  $Q_j'(1) \neq 1$  and therefore such  $\lambda_j$  exist. The second statement follows directly from the definition of  $\mathbf{qMZV}$ .  $\square$

The following proposition allows one to write an arbitrary element in  $Z(Q, \mathbb{N}_{>1})$  as an linear combination of  $[s_1, \dots, s_l] \in \mathcal{MD}^\#$ .

**Proposition 2.5.** Assume  $k \geq 2$ . For  $1 \leq i, j \leq k-1$  define the numbers  $b_{i,j}^k \in \mathbb{Q}$  by

$$\sum_{j=1}^{k-1} \frac{b_{i,j}^k}{j!} t^j := \binom{t+k-1-i}{k-1} .$$

With this it is for  $1 \leq i \leq k-1$  and  $Q_j^E(t) = \frac{1}{(j-1)!} t P_j(t)$

$$t^i = \sum_{j=2}^k b_{i,j-1}^k (1-t)^{k-j} Q_j^E(t) .$$

**Proof.** We want to show that

$$\frac{t^i}{(1-t)^k} = \sum_{j=1}^{k-1} \frac{b_{i,j}^k}{j!} \frac{tP_j(t)}{(1-t)^{j+1}}$$

By the definition of the Eulerian Polynomials it is

$$\begin{aligned} \sum_{j=1}^{k-1} \frac{b_{i,j}^k}{j!} \frac{tP_j(t)}{(1-t)^{j+1}} &= \sum_{j=1}^{k-1} \frac{b_{i,j}^k}{j!} \sum_{d>0} d^j t^d \\ &= \sum_{d>0} \left( \sum_{j=1}^{k-1} \frac{b_{i,j}^k}{j} d^j \right) t^d \\ &= \sum_{d>0} \binom{d-i+k-1}{k-1} t^d \end{aligned}$$

The claim now follows directly from the easy to prove formula

$$\frac{1}{(1-t)^k} = \sum_{n \geq 0} \binom{n+k-1}{k-1} t^n.$$

□

We give some examples how to write elements in  $\mathbf{qMZV}$  as linear combinations of elements in  $\mathcal{MD}$ . From the proposition we deduce for the length one case for all  $k > 0$

$$Z(2k) = \sum_{j=2}^{2k} b_{k,j-1}^{2k} [j] \quad \text{and} \quad Z(2k+1) = \sum_{j=2}^{2k+1} (b_{k,j-1}^{2k+1} + b_{k+1,j-1}^{2k+1}) [j].$$

Clearly this also suffices to give linear combinations in higher length.

**Example 2.6.** We give some examples

$$\begin{aligned} Z(2) &= [2], & Z(3) &= 2[3], \\ Z(4) &= [4] - \frac{1}{6}[2], & Z(5) &= 2[5] - \frac{1}{6}[3], \\ Z(6) &= [6] - \frac{1}{4}[4] + \frac{1}{30}[2], & Z(7) &= 2[7] - \frac{1}{3}[5] + \frac{1}{45}[3], \\ Z(2,2) &= [2,2], & Z(2,4) &= [2,4] - \frac{1}{6}[2,2]. \end{aligned}$$

The  $q$ -expansion of modular forms are well known to give rise to  $q$ -analogues of Riemann zeta values. Let us denote by  $M_{\mathbb{Q}} = \mathbb{Q}[G_4, G_6]$  and  $\bar{M}_{\mathbb{Q}} = \mathbb{Q}[G_2, G_4, G_6]$  the ring of modular and quasi-modular forms, where the Eisenstein series  $G_2$ ,  $G_4$  and  $G_6$  are given by

$$G_2 = -\frac{1}{24} + [2], \quad G_4 = \frac{1}{1440} + [4], \quad G_6 = -\frac{1}{60480} + [6].$$

We clearly have the following inclusions of  $\mathbb{Q}$ -algebras

$$M_{\mathbb{Q}} \subset \widetilde{M}_{\mathbb{Q}} \subset \mathfrak{qMZV} \subset \mathcal{MD}.$$

where the second inclusion follows from

$$\begin{aligned} G_2 &= -\frac{1}{24} + Z(2), \\ G_4 &= \frac{1}{1440} + Z(2) + \frac{1}{6}Z(4), \\ G_6 &= -\frac{1}{60480} + Z(6) + \frac{1}{4}Z(4) + \frac{1}{120}Z(2). \end{aligned}$$

In the theory of modular forms the operator  $d := q \frac{d}{dq}$  plays an important role and it is a well known fact that  $\widetilde{M}_{\mathbb{Q}}$  is closed under  $d$ . In [BK] the authors showed the following

**Theorem 2.7.** The operator  $d$  is a derivation on  $\mathcal{MD}$  that is compatible with the filtrations on  $\mathcal{MD}$  given by the weight and the length.

In [Ok] the following conjecture is stated by Okounkov

**Conjecture 2.8.** The operator  $d$  is a derivation on  $\mathfrak{qMZV}$ .

For the derivative of a length one generating series of multiple divisor sums we have several identities. These will be used to make the following result which gives some evidence for the conjecture above.

**Proposition 2.9.** It is  $dZ(k) \in \mathfrak{qMZV}$  for all  $k \geq 2$ .

**Proof.** In [BK] Theorem 3.5 the authors prove the following representation of the derivative  $d[k-2]$

$$\begin{aligned} \binom{k-2}{s_1-1} \frac{d[k-2]}{k-2} &= [s_1] \cdot [s_2] - [s_1, s_2] - [s_2, s_1] \\ &+ \binom{k-2}{s_1-1} [k-1] - \sum_{\substack{a+b=k+2 \\ a>s_1}} \left( \binom{a-1}{s_1-1} + \binom{a-1}{s_2-1} - \delta_{a,s_2} \right) [a, b]. \end{aligned}$$

where  $s_1, s_2 > 0$  can be chosen arbitrary such that  $k = s_1 + s_2$ . First divide both sides by  $\binom{k-2}{s_1-1} (k-2)^{-1}$ . Whenever  $k \geq 4$  all elements on the right of the resulting equation belong to  $\mathcal{MD}^\sharp$  except for the term with  $[k-1, 1]$ . By direct calculation one obtains that for  $s_1 = 1$  and  $s_2 = k-1$  the coefficient of  $[k-1, 1]$  is  $-(k-2)$  and for  $s_2 = 2$  and  $s_1 = k-2$  it is  $-2(k-2)$  and therefore  $d[k-2]$  can be expressed as an element in  $\mathcal{MD}^\sharp$ .  $\square$

Since  $d$  is a derivation it satisfies the Leibniz rule. Therefore the above proposition allows us to derive further identities, e.g.

$$dZ(k, \dots, k), d(Z(k_1, k_2) + Z(k_2, k_1)) \in \mathfrak{qMZV}.$$

**Example 2.10.** Some examples of representations of  $dZ(s)$  in  $qMZV$ .

$$\begin{aligned}
dZ(2) &= 3Z(4) + Z(2) - Z(2, 2), \\
dZ(3) &= 5Z(5) + Z(3) - 4Z(3, 2) - 6Z(2, 3), \\
dZ(4) &= 10Z(6) + 2Z(4) + 4Z(4, 2) - 8Z(2, 4) - 6Z(3, 3), \\
dZ(2, 2) &= -6Z(6) - 12Z(2, 2, 2) - 15Z(4, 2) + 3Z(2, 4) + 9Z(3, 3), \\
dZ(3, 3) &= 4Z(8) - 12Z(2, 3, 3) - 10Z(3, 2, 3) - 8Z(3, 3, 2) \\
&\quad + Z(3, 5) - Z(5, 3) + 8Z(6, 2) + 3Z(3, 3), \\
dZ(2, 2, 2) &= -24Z(2, 2, 2, 2) + 9Z(2, 3, 3) + 9Z(3, 2, 3) + 6Z(3, 3, 2) \\
&\quad - 15Z(4, 2, 2) - 15Z(2, 4, 2) + 3Z(2, 2, 4) - 6Z(2, 6) + 6Z(5, 3) - 6Z(6, 2).
\end{aligned}$$

At the end we give some conjectured representations of  $dZ(s)$  in  $qMZV$  coming from numerical experiments and which were checked for the first 200 coefficients but which should be also provable by using the results in [BK].

$$\begin{aligned}
dZ(2, 3) &= 2Z(7) - 16Z(2, 2, 3) - 4Z(2, 3, 2) - 8Z(3, 2, 2) \\
&\quad - 15Z(4, 3) - 4Z(3, 4) + 4Z(5, 2) + 5Z(2, 5) + Z(3, 2) - Z(2, 3),
\end{aligned}$$

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